Almost Periodicity in Solid State Physics and C*Algebras

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I. *Almost Periodic Physics:*

Several physical phenomena involve almost or quasi periodic functions. The earliest examples concerned applications in Classical Mechanics. More recently almost periodicity has been important in Quantum Mechanics especially in problems involving conductors. Most of the corresponding examples concern Schrödinger operators with quasi or almost periodic potential or some tight binding approximation of it. The aim of this section is to provide physical examples taken from Solid State Physics.

I-1. *Quasi 1D conductors:*

In 1964 Little [Little], in a remarked article suggested that superconductivity could be enhanced in organic conductors. More generally, molecular conductors represent a favourable case for such a mechanism because they may contain easily 20 to 40 time more atoms than a metal in a unit cell and the intermolecular vibrations permit an increase of the interactions between Cooper pairs. These remarks led the community to search for conducting organic crystals. In the early seventies the salts of TTF (tetrathiofulvalene) were produced in particular the TTF-TCNQ. The corresponding molecules are planar and are vertically linked together through hydrogen bridges leading to a very strong anisotropy and also to the existence of a conduction band in the vertical direction. It was soon realized however that most of them even though quite good conductors at room temperature, became insulator at low temperature preventing a superconductor transition to occur. In 1979 Jerome, Bechgaard et al. [Schultz] found a new family of molecules, similar to the TTF salt, the so called TMTSF salts (tetramethyl-tetraselena-fulvalene) providing a superconductor transition at low temperature. Our aim here is not to consider the superconductor transition but rather to provide an explanation for the existence of a metal-insulator transition in the early examples.

In describing the metallic properties of such a chain, one usually ignores the electron interaction, and the only collective constraints comes from Pauli's principle leading to the Fermi-Dirac distribution at thermal equilibrium. It is then sufficient to investigate the one electron Hamiltonian. In our problem since the conductivity is essentially one dimensional, it will be sufficient to represent it as a 1D Schrödinger operator. Thanks to the periodic arrangement of the molecules, the effective potential V seen by a typical conduction electron will be a spatially periodic function of a period "a" determined by the chemical forces. The Bloch theory, the Solid State analog of Floquet's theory, predicts that the energy spectrum is obtained by searching eigenfunctions satisfying Bloch's boundary conditions namely, in suitable units:

$$\{ -\frac{\partial^2}{\partial x^2} + V(x)\} \ \psi_k(x) = E(k)\ \psi_k(x) \qquad \psi_k(x+a) = e^{ika}\ \psi_k(x) \tag{1}$$

The electron gas will then occupy all energy levels below the chemical potential which usually coincides at low temperature with the Fermi energy level E_F. However in these systems, because the chemical bonds are not as strong as in metals, the electron gas has another possibility to decrease its overall energy, namely by creating a gap at the energy level (fig. 1). This is called the "Peierls instability" [Peierls (55)]. It is obtained through

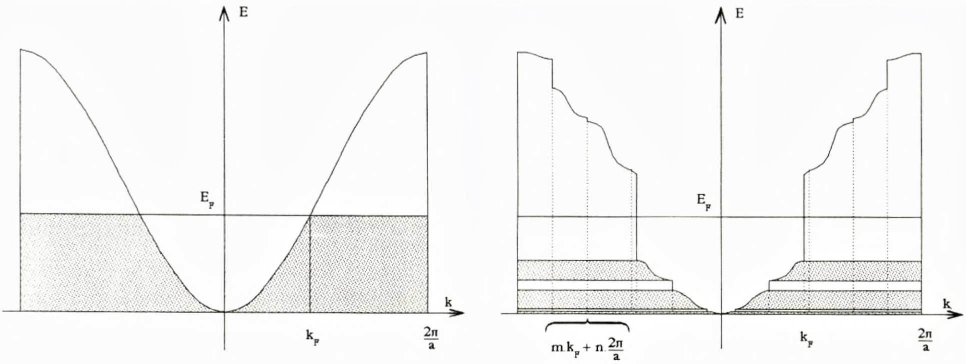

Fig. 1: 1) left: in absence of spatial modulation of the charge density the electron gas occupies the states with energy below the Fermi level.

2) right: if the charge density modulated itself spontaneously, a gap opens at the Fermi level, decreasing the overall energy of the electron gas. This modulation of the CDW is therefore stable (Peierls instability).

a modulation of the electron gas at a spatial frequency $a_F = 2\pi/k_F$ where k_F is the quasi momentum such that $E(k_F) = E_F$. Actually the modulation usually affects the "charge density wave" (CDW), namely the charge distribution in the electron gas along the chain. This effect creates an additional contribution to the effective potential with a spatial period a_F. *Since in general a_F is not commensurate to a the effective potential seen by the conduction electrons is quasi periodic.* Aubry [Aubry 78] proposed, to describe this phenomenon, the following tight binding model, called the *Almost Mathieu equation:*

$$\phi(n+1) + \phi(n-1) + 2\mu\cos 2\pi(x-\alpha n)\ \phi(n) = E\ \phi(n) \tag{2}$$

In this equation, μ represents the strength of the interaction, $\alpha = a_F/a$ is the frequency ratio, and x is a random phase representing the arbitrariness of the origin in the crystal (phason modes). We will see later on in this review that indeed if the extra modulation is strong enough, the corresponding quasi periodic Schödinger operator has a pure point spectrum at low energy leading to exponentially localized states and zero conductivity. It is therefore not surprising to find in general a metal insulator transition at low temperature for these systems. What makes the difference between various molecules is the strength of the Peierls instability. In the TMTSF salts, it seems to be weak enough to avoid the insulator state, and therefore to permit at low temperature the creation of Cooper pairs leading to superconductivity.

I-2. *2D Bloch electrons in a uniform magnetic field:*

The second example of a system described by a quasi periodic potential concerns an electron gas in a two dimensional perfect crystal submitted to a uniform perpendicular magnetic field. This problem has been one of the most challenging encountered by Solid State Physicists. The first proposal to treat it goes back to the thirties with the works of Landau [Landau (30)] and Peierls [Peierls (33)], who gave the lowest order approximation of the effective hamiltonian at respectively high and low magnetic field. The question of finding an accurate effective hamiltonian occupied most of the experts during the fifties (see [Bellissard (88a)] for a short review of that question). The main reason comes from the usefullness of the magnetic field in providing efficient experimental tools for measuring microscopic properties of metals. The Hall effect, the de Haas-van Alfven oscillations, the magnetoresistance, for example provide precise information on the charge carriers, the shape of the Fermi surface, the band spectrum, etc. During the sixties and the seventies, many improvements were obtained on the nature of the corresponding energy spectrum. In particular D. R. Hofstadter computed the spectrum of the so called Harper model as a function of the magnetic flux through a unit cell, end exhibited an amazing fractal structure (see fig. 2) which is still now under

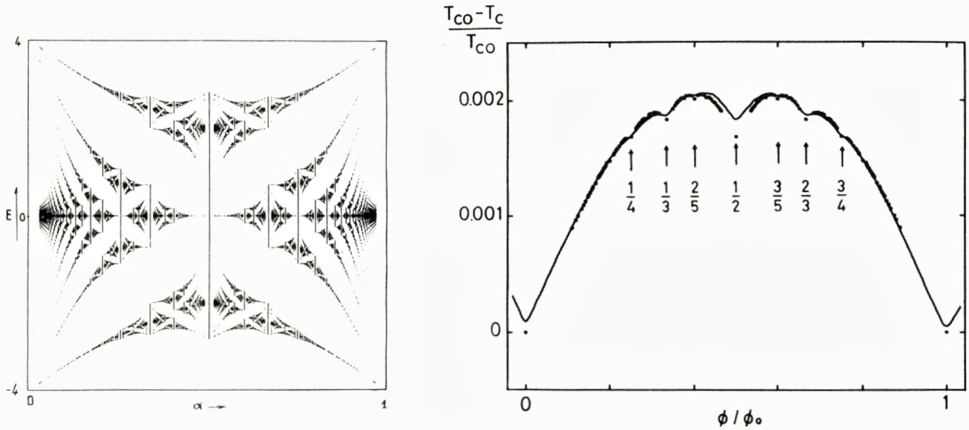

Fig. 2: 1) left: the Hofstadter spectrum as a function of the parameter α.
2) right: measurement of the transition curve between normal and superconduction phase in the (T,B) plane for a square network of filamentary superconductors (taken from [Pannetier (84)]).

study, even though recent results permit to say a lot on it (see [Bellissard (88b)] for a review).

In order to give an idea of how quasi periodicity enters in this game let us consider a rather simple example. Let us assume that an electron be described in a tight binding approximation, by a wave function ψ on a 2D square lattice $\mathbf{Z}^2$.In absence of magnetic field, the energy operator may be effectively described, as a first approximation, by means of nearest neighbours interaction, namely by

$$H\psi\,(m,n) = \psi(m+1,n) + \psi(m-1,n) + \psi(m,n+1) + \psi(m,n-1). \tag{3a}$$

Adding a uniform magnetic field will result in adding a U(1) gauge field, namely in changing the phase of each therm in (2):

$$\begin{aligned} H(B)\ \psi\ (m,n) = \quad & e^{2i\pi A_1(m,n)}\ \psi(m+1,n) + e^{-2i\pi A_1(m-1,n)}\ \psi(m-1,n) \\ & + e^{2i\pi A_2(m,n)}\ \psi(m,n+1) + e^{-2i\pi A_2(m,n-1)}\ \psi(m,n-1) \end{aligned} \tag{3b}$$

where $A_\mu(m,n)$ represents the product of e/h (h being the Planck constant) by the line integral of the vector potential between the point (m,n) of the lattice and the point $(m+1,n)$ for $\mu=1$, or $(m,n+1)$ if $\mu=2$. In particular, because the magnetic field is uniform, one must have:

$$A_1(m,n) + A_2(m+1,n) - A_I(m,n+1) - A_2(m,n) = \frac{\phi}{\phi_0} = \alpha \tag{4}$$

where $\phi_0 = h/e$ is the quantum of flux and ϕ the flux through the unit cell. One solution of the previous equation (4) is the "Landau gauge" namely:

$$A_1(m,n) = 0 \qquad\qquad A_2(m,n) = \alpha m \tag{5}$$

In this case, the operator $H(B)$ commutes with space translations along the n-direction. Therefore the solutions of the stationary Schrödinger equation will have the form:

$$H(B)\ \psi = E\ \psi \qquad \text{with} \qquad \psi(m,n) = e^{-2i\pi kn}\ \phi(n) \tag{6}$$

leading to Harper's equation:

$$\phi(n+1) + \phi(n-1) + 2\cos 2\pi(k-\alpha n)\ \phi(n) = E\ \phi(n) \tag{7}$$

Thanks to eq. (4) "α" is a physical parameter liable to vary, and will be therefore irrational most of the time. The Harper equation appears as a discrete version of a 1D Schrödinger operator with a quasi periodic potential.

If the crystal is not a square lattice but a rectangular one, leading to anisotropy between the two components, the same argument leads to the Almost Mathieu equation:

$$\phi(n+1) + \phi(n-1) + 2\mu\cos 2\pi(k-\alpha n)\ \phi(n) = E\ \phi(n) \tag{7}$$

where μ represents the anisotropy ratio of the coupling constants in the vertical versus the horizontal directions. This equation also represents the effective hamiltonian for a tight binding representation of the effect of a Charge Density Wave in a 1D conductor provided μ represents the strength of the Peierls instability (see eq. (2)).

It is important to remark that (3b), (6) or (7) can be written in an algebraic way by introducing the following two unitaries U and V:

$$U\,\psi\,(m,n) = e^{-2i\pi A_1(m-1,n)}\ \psi(m-1,n) \qquad V\,\psi\,(m,n) = e^{-2i\pi A_2(m,n-1)}\ \psi\,(m,n-1) \tag{8}$$

They satisfy the following commutation relation:

$$U\,V = e^{2i\pi\alpha}\ V\,U \tag{9}$$

The Almost Mathieu hamiltonian can be written as:

$$H = U + U^* + \mu(V + V^*) \tag{10}$$

and in general it is possible to show (see §III.1) that the band hamiltonian for a 2D Bloch electron in a uniform magnetic field belongs to the C*Algebra generated by U and V.

I-3. *Superconductor networks:*

In the Landau-Ginzburg approach [Landau (50)] of the superconductivity, the state of the electron gas is represented by a unique coherent wave function $\Psi(x)$. It plays the role of an order parameter like the magnetization in magnetic systems. The square $|\Psi(x)|^2$ of this wave function will represent phenomenologically the probability density of Cooper pairs in a sort of Hartree approximation. Landau and Ginzburg postulated that the corresponding free energy is given by:

$$F = \int_{\Sigma} d^3\mathbf{x}\ \{|(\frac{h}{2\mathrm{i}\pi}\partial - 2e\mathbf{A}(\mathbf{x}))\ \Psi(\mathbf{x})\ |^2 + \alpha\,|\,\Psi(\mathbf{x})|^2 + \beta\,|\,\Psi(\mathbf{x})|^4 + \frac{|\mathbf{H}(\mathbf{x})|^2}{8\pi}\} \tag{11}$$

where Σ is the volume occupied by the superconductor, e the charge of the electron, ∂ the gradient operator, **A** the vector potential, **H** the effective magnetic field in the bulk, and α, β are temperature dependent phenomenological parameters. To insure the stability of the system, we must have $\beta > 0$. The actual state of the system is provided by functions minimizing the free energy. Since at temperature bigger than the critical temperature there is no Cooper pairs, one must assume that the minimum is reached for $\Psi=0$. This implies in turn that α is positive for $T > T_c$. If $T < T_c$, we must have a non zero solution, and therefore $\alpha < 0$. Assuming a smooth dependence in the temperature, we get:

$$\alpha(T) \approx (T - T_c)\left(\frac{d\alpha}{dT}\right)_{T_c} \qquad \beta(T) \approx \beta_c \qquad at \quad T \approx T_c \tag{12}$$

In a large superconductor, the magnetic field does not penetrate in the bulk (Meissner effect), unless under the form of quantized flux tubes [Mermin]. The penetration length $\xi(T)$ can be computed in terms of the parameters α and β and is of order of about 1000Å at small T's. This can be seen by computing the minimizing solution of (11) for a half space for instance [Landau (50), Jones]. Near the critical temperature however the penetration length diverges like $\xi(T) \approx \xi_0\,(1-T/T_c)^{-1/2}$, and Ψ must be very small, in such a way that the quartic term in (11) may be neglected. Therefore whenever the external magnetic field is uniform, for superconductors of small size, the minimizing solution of (11) is such that **H** $\approx$ const. in the bulk and Ψ satisfies the linearized equation:

$$\left\{\frac{h}{2\mathbf{i}\pi}\partial - 2e\mathbf{A}(\mathbf{x})\right\}^2 \Psi(\mathbf{x}) = E\Psi(\mathbf{x}) \qquad E = \left(\frac{d\alpha}{dT}\right)_{T_c}(T_c - T) \tag{13}$$

with some proper boundary condition. To get the minimum of the free energy, E *must be the groundstate of* (13).

These remarks were the basic elements for the study of filamentary superconductors as initiated by DeGennes [deGennes (81)] and Alexander [Alexander], in the study of random mixtures of superconductors and normal metals. The solution of (13) for a thin filament of finite length can be obtained through the one dimensional analog of (13) and a gauge transformation. It is therefore sufficient to know the wave function at the filament ends to know the solution everywhere. The compatibility conditions (current conservation) at the filaments edges give rise to a sort of tight binding representation of the linearized Landau-Ginzburg equation (13). For regular lattices of filamentary superconductors these equations have been written by Alexander, Rammal, Lubensky and Toulouse [Rammal (83)]. For a square lattice of infinitely thin filaments of length "a" one gets:

$$H(B)\ \psi = \varepsilon\ \psi \qquad \varepsilon = 2\cos(a.(E)^{1/2}) = 2\cos\ (a/\xi(T)) \tag{14}$$

where ψ represents the sequence of values of Ψ at the vertices of the lattice, $H(B)$ is the operator given by eq. (3) provided the electron charge e be replaced by the charge of a Cooper pair $2e$ and ε be the groundstate energy of $H(B)$. For real filaments, the thickness is usually not small enough, and a correction due to the bulk must be introduced to fit the experiments.

Eventually the Grenoble group (Chaussy, Pannetier, Rammal and coworkers) performed an experiment on a hexagonal lattice [Pannetier (83)] and a square lattice [Pannetier (84)]: they measured the field dependence of the critical temperature, which is related through (13) to the corresponding groundstate energy of the linearized Landau-Ginzburg equation. The calculation of ε is quite easy numerically and the comparison with the experiment is amazingly accurate (fig. 2). Not only do we get a flux quantization at integer multiples of ϕ_0 ($\phi_0 = h/2e$) but also at fractional values, exactly like in the Hofstadter spectrum. Later on the experiment has been performed on a Penrose lattice, a quasi periodic one [Behrooz], and also on a Sierpinsky gasket [Ghez].

More recently, the Grenoble group realized that the measurement of the magnetic susceptibility near the critical line is related to the derivate of ε with respect to flux ϕ/ϕ_0 thanks to the Abrikosov theory of type II superconductors [Abrikosov]. The Wilkinson-Rammal formula (see [Bellissard (88b)] and section III below) permits to compute this derivative at each rational value of ϕ/ϕ_0. Again the comparison with the experiment is amazingly accurate [Gandit]. The magnetic susceptibility admits a discontinuity at each rational value of ϕ/ϕ_0 in agreement with the Wilkinson-Rammal formula. To date this is the only experiment where these quantities about the Hofstadter spectrum, can be measured so accurately.

I-4. *Normal Conductor networks:*

In a normal metal, one usually explains the weak localization by the existence of an interference increasing the backscattering [Bergmann]. More precisely, due to the slight disorder in the metal, one considers the electron wave as scattered by the randomly distributed impurities. In this process, considering a diffusion path $O, A_1, A_2, \ldots, A_n, O'$ the averaging over the positions $A_1, A_2, \ldots,$ of the scatterers usually decreases the sum of the diffusion amplitude distroying all interference. However, if $O = O'$ (backscattering), the waves following the path forward $O, A_1, A_2, \ldots A_n, O$ and backward $O, A_n, A_{n-1}, \ldots, A_1, O$ have no phase difference and they always interfere whatever the position of the scatterers. Thus the backscattering amplitude is higher than the forward scattering, decreasing the electric conductivity.

This effect however occurs as long as the time reversal symmetry is not broken. Adding a magnetic field will decrease the backscattering and the magnetoresistance as well. The phase shift between the two forward and backward paths will be given by

$2\pi e\phi/h$ for each of these paths, namely $2\pi 2e\phi/h = 2\pi\phi/\phi_0$ with now $\phi_0 = h/2e$. Thus as for superconducting systems the effective charge is $2e$ instead but the mechanism is completely different.

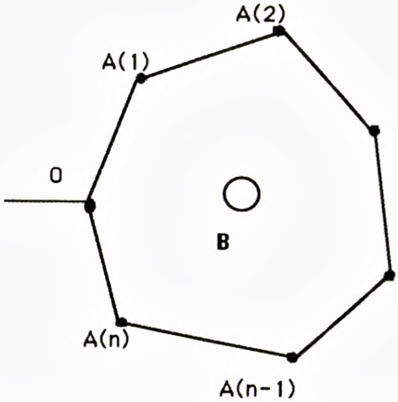

Fig. 3: A typical diffusion path for a quantum wave. The phase shift between the path O, A(1),..., A(n), O and the path O, A(n),..., A(1), O is $2\pi\phi/\phi_0$ where $\phi_0 = h/2e$. The replacement of e by 2e comes from the weak localization effect and not from the existence of pairs as in the theory of superconductivity.

The computation of the conductivity is always tricky, in order to take into account the collision time and the phase coherence time. But this weak localization approach gives rise to a correction $\delta\sigma$ to the conductivity given by [Bergmann, Douçot (85)& (86)]:

$$\delta\sigma(x) = -2/\pi\, e^2/h\; C(x,x) \tag{15}$$

where $C(x,x')$ is the Green function defined as the solution of:

$$\left\{ \left(-\mathbf{i}\partial - \frac{2\pi 2e}{h}\mathbf{A}(\mathbf{x})\right)^2 + \frac{1}{L_\phi^2} \right\} C(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}') \tag{16}$$

In this formula, L_ϕ represents the phase coherence length. $C(x,x)$ is usually called the "Cooperon".

One way to investigate this effect consists in looking at a filamentary conductor in which there are loops. If L is the typical loop size, and l the mean free path, one must have $l \ll L$, in order to get weak localization results, but $L \leq L_\phi$ if one wants to observe the interference effect. In this case, the magnetoresistance must be the same for each magnetic field such that the flux through the loop is an integer multiple of ϕ_0. These oscillations have been observed first by Sharvin and Sharvin [Sharvin] on a simple loop and the phenomena are enhanced for a regular lattice of thin wires. Treating eq. 16 as the Landau-Ginzburg equation for a lattice of filamentary superconductors, we eventually obtain the Cooperon from the value of the Green function of the Harper hamiltonian $H(B)$. The actual formula for the resistance of a regular 2D lattice was computed by Douçot and Rammal [Douçot (85) & (86a)] and is given by:

$$\frac{\Delta R}{R} = \kappa \left\{ \frac{\eta ch\eta - sh\eta}{\eta sh\eta} \left(1 - \frac{2}{Z}\right) + 2\, sh\eta\, \tau \left(\frac{\mathbf{1}}{4ch\eta - H(B)} \right) \right\} \tag{17}$$

$$\eta = \frac{a}{L_\phi} \qquad\qquad \kappa = \frac{2e^2 L_\phi}{\sigma_0 hS} \tag{18}$$

where Z is the coordination number of the lattice, "a" is the lattice spacing, S is the cross section of the wires, σ_0 is the conductivity of the corresponding perfect conductor as computed by neglecting the weak localization effect, and $H(B)$ is the Harper operator for the corresponding lattice (for a square lattice see eq. 3). In the formula (17), τ represents the trace per unit volume of the operator in parenthesis (see §III).

The measurement of such a resistance has been performed again by the Grenoble group [Douçot (85) (86b)] and the comparison with the experiment is also amazingly good. This is a spectacular confirmation of the validity of weak localization theory.

I-5. *Quasicrystals:*

In 1984, Schechtman, Blech, Gratias, Cahn [Schechtman] found a new kind of crystalline order in an Al-Mn alloy giving rise to a perfect X-ray diffraction pattern with a five-fold symmetry. Since it is well known that no cristalline group in 3D exists with a five-fold symmetry axis [Mermin], they were led to admit that the translation invariance was broken. Nevertheless because of the quality of the diffraction picture, they proposed a quasi periodicity atomic arrangement. In the early seventies, Penrose [Penrose] had produced examples of quasi periodic tillings of the plane, leading to examples with a five-fold symmetry axis. A systematic rigorous framework of his ideas was proposed by de Bruijn [de Bruijn] and new constructions permitted to produce such arrangements in 2D and 3D. One construction consists in projecting a higher dimensional regular lattice onto a 2D or 3D linear subspace with incommensurate slopes. The icosahedral symmetry observed in the original samples, is realized in $\mathbf{Z}^6$, supporting a representation of the icosahedral group [Duneau, Kramer]. This representation can then be decomposed into a direct sum of two irreducible representations of dimension 3 corresponding to subspaces denoted by E_+ and E_-. To get an example of a quasiperiodic lattice the strip method consists in considering the "strip" Σ obtained by translating the unit semi open cube $[0,1)^{\times 6}$ in $\mathbf{R}^6$ along the E_+ directions, and in projecting all points in $\mathbf{Z}^6 \cap \Sigma$ on E_+ along the E_- direction. If now E_+ is identified with $\mathbf{R}^3$ one gets a sublattice in 3D invariant by the icosahedral group which is obviously quasi periodic by construction. Moreover it can be shown that such a structure is also invariant by a discrete group of dilations generated by some power of the golden mean. This last fact is not so surprising since the golden mean is related to the cosine of $2\pi/5$. If one represents the sites of this lattice by means of the sum of Dirac measures located at each site, the diffraction pattern obtained by taking the Fourier transform of this measure coincides in position and also rather well in intensity with the experimental observation [Gratias]. Other kinds of quasicrystals have been observed with ten-fold, twelve-fold, and more recently eight-fold symmetries [Kuo] giving rise to a new area in crystallography, called "non Haüyan" in contrast to the standard theory originally formulated by Haüy.

Nevertheless we will have eventually to understand the electronic or mechanical properties of such structures. The phonon spectrum, namely the distribution of the vibrational modes is needed to compute the heat capacity of the thermal conductivity of the quasicrystal. The electron spectrum will help in computing the electric conductivity. Unfortunately quasi periodic Schrödinger operators in more than one dimension are not yet understood. This is probably the reason why most of the models investigated up to now are one dimensional. The strip construction in one dimension from $\mathbf{Z}^2$ leads to a chain of points x_n on the real lines such that $x_{n+1}-x_n$ takes on two incommensurate values distributed in a quasiperiodic way. The phonon spectrum for such a crystal can be described by the spectrum of the following discrete Schrödinger equation [Luck]:

$$\psi(n+1) + \psi(n-1) + \lambda\, \chi_A\, (x-n\alpha)\; \psi(n) = E\; \psi(n) \tag{19}$$

where χ represents the characteristic function of the interval A of the unit circle, x is a random phase defined modulo 1 and α is an irrational number. It turns out that the spectral properties of this family of equations are fairly different from the properties of the Harper or Almost Mathieu equations. As was proved by Delyon and Petritis [Delyon (86)], for a large set of α's (19) has no eigenfunctions converging to zero at infinity. Moreover, an argument due to Kadanoff, Kohmoto and Tang [Kadanoff], and Ostlund [Ostlund (83)] supplemented by rigorous proofs of Sütö [Sütö] and Casdagli [Casdagli], shows that for α the golden mean, and λ big enough, the spectrum is a Cantor set of zero Lebesgue measure and non-zero Hausdorff dimension. In particular the spectral measure is singular continuous. The spectrum of (19) as a function of α has been computed numerically by Ostlund and Pandit [Ostlund] and has a simpler structure than the Hofstadter spectrum (fig. 2). This work suggests that the spectrum is a Cantor set of zero Lebesgue measure for any irrational α's. The corresponding eigenstates for α the golden mean were partially computed by Kadanoff, Kohmoto and Tang and also by Ostlund et al. [Ostlund] and exhibit strong recurrence properties in space, being localized around an infinite sequence of points, a result which looks like intermittency. In other words if the wave function is interpreted as the amplitude of the lattice excitation in the crystal, there is an infinite sequence of clusters of atoms far away from each other, in which the lattice oscillations are big whereas the other atoms are essentially at rest.

The corresponding two dimensional model on a Penrose lattice has been studied numerically by Kohmoto and Sutherland [Kohmoto], and is likely to provide also a Cantor spectrum with spatial intermittency. They have discovered also the existence of infinitely degenerate eigenvalues with eigenstates localized in a bounded region (molecular states), like in the case of a Sierpinski gasket [Rammal (84)]. However essentially nothing is known on the nature of the spectrum.

II. *Schrödinger Operators with Almost Periodic Potential:*

In this section we consider a Schrödinger operator H on $\mathbf{R}^D$ (continuous case) or $\mathbf{Z}^D$ (discrete case) with $D=1$ in most cases and some indications for $D \geq 2$, namely:

$$H\,\psi(x) = -\Delta\psi(x) + V(x)\psi(x) \qquad \psi(x) \in L^2(\mathbf{R}^D)$$

or (1)

$$H\,\psi(x) = -\Sigma_{|e|=1}\,\psi(x-e) + V(x)\psi(x) \qquad \psi(x) \in l^2(\mathbf{Z}^D)$$

where V is almost periodic on $\mathbf{R}^D$ or on $\mathbf{Z}^D$.

These operators exhibit three kinds of properties:

- they tend to have nowhere dense spectra. But it is only a generic property in general; counter examples are known.
- if V is sufficiently smooth, they have a tendency to exhibit a transition between an absolutely continuous and a pure point spectrum when the coupling constant is increased. This is interpreted physically as a metal-insulator transition. In most cases investigated, the eigenfunctions corresponding to the absolutely continuous component are Bloch waves whereas the eigenstates of the pure point spectrum are exponentially localized.
- if the potential is not smooth, if the frequency module is not diophantine or if the coupling constant is critical, the spectrum has a tendency to be singular continuous.

II-1. *Nowhere dense spectra:*

Historically, one of the first rigorous results concerning the gaps of a Schrödinger operator with a quasi periodic potential was provided by Dubrovin, Matveev and Novikov [Dubrovin]. Investigating quasi periodic solutions of the KdV equation by means of the inverse scattering method, they were able to construct a class of potentials for the 1D case giving rise to a spectrum having finitely many gaps. This result has been recently extended by a work of Johnson and de Concini [Johnson] for the case of infinitely many gaps having some regularity property. This family of potentials is obtained by constructing a Jacobi surface depending on the spectrum and a canonical torus associated to that surface, in such a way that V is the restriction of an algebraic function on this torus to the orbit of a constant vector field on this torus. This is the reason why such a potential is called "algebraic-geometric". They constitute a family with a finite number of parameters and for this reason it is non generic in the space of almost or even quasi periodic functions with the same frequency module.

THEOREM 1: The set of almost periodic finite zone potentials with a spectrum given by $\Gamma = \bigcup_{i\in[0,N]}[E_{2i}, E_{2i+1}]$ with $E_{2N+1}=\infty$ is isomorphic to the Jacobian variety $J(\Gamma)$ (namely a 2N-torus) of the Riemann surface $R(\Gamma)= \{(W,E)\in\mathbf{C}^2\,;\, W^2-P_{2N+1}(E)=0\}$ if $P_{2n+1}(E)$ is the polynomial $\Pi_{i\in[0,N]}(E-E_i)$.

◊

In 1980, J. Moser [Moser], J. Avron and B. Simon [Avron (81)] and Chulaevski [Chulaevski] proved a result concerning the generic character of nowhere dense spectra. A limit periodic function f is a continuous function on $\mathbf{R}$ which is a uniform limit of a sequence $\{f_n\}$ of continuous periodic functions on $\mathbf{R}$. If T_n is the period of f_n we must have $T_{n+1}/T_n \in \mathbf{N}$. The same definition applies for limit periodic sequences. Let L be any separable Fréchet topological vector space of limit periodic functions or sequences, we get:

THEOREM 2: If $D=1$, there is a dense G_δ set $L°$ in L such that if $V\in L°$, the operator H in (1) has a nowhere dense spectrum.

◊

An interesting class of the limit periodic models giving rise to a nowhere dense spectrum, is given by Jacobi matrices of a Julia sets. The first example was provided by Bellissard, Bessis and Moussa [Bellissard (82d)], and concerned polynomials of degree 2. Their work was extended to polynomials of higher degree by Barnsley, Geronimo and Harrington [Barnsley (83,85)]. Let P be a polynomial of degree N with real coefficients. One will assume that P is monic, namely $P(z)=z^N+O(z^{N-1})$. One considers the dynamic on the complex plane $\mathbf{C}$ defined by:

$$z(n+1) = P(z(n)) \tag{2}$$

In general it has finitely many attracting fixpoints including the point at infinity, each having an open basin of attraction. The Julia set $J(P)$ of P is the complement of the union of them. It is a compact set. The Fatou-Julia theorem [Fatou, Julia] gives necessary and sufficient condition in order that $J(P)$ be contained in the real line and completely disconnected. When this happens, there is a unique probability measure μ_p on $J(P)$ called the balanced measure, which is both P and P^{-1} invariant. It is singular continuous. The general theory of orthogonal polynomials allows to associate canonically to μ_p a Jacobi matrix (namely an infinite tridiagonal matrix indexed by $\mathbf{N}$) in the following way: consider in $L^2(J(P), \mu_p)$ the orthogonal basis p_n obtained from the set of monomial functions $x \in J(P) -> x^n$, $n \in \mathbf{N}$, by the Gram-Schmidt process. It is easy to show that p_n is a monic polynomial of degree n, such that $p_n(P(x)) = p_{nN}(x)$ and $p_0 = 1$. The Jacobi matrix $H(P)$ associated to P is the matrix operator of multiplication by x in $L^2(J(P), \mu_p)$, in the previous basis properly normalized. Since $J(P)$ is compact it follows that $H(P)$ is a bounded operator. By construction, $J(P)$ is the spectrum of $H(P)$ and μ_p is equivalent to its spectral measure. Therefore we get a class of self-adjoint operators having a singular continuous spectrum. The remarkable property of this class lies in the following remark. Let D be the operator on $L^2(J(P), \mu_p)$ defined by $D^*f(x) = f(P(x))$. Due to the invariance properties of the balanced measure, it is easy to see that D is a partial isometry such that [Bellissard (85b)]:

$$DD^* = \mathbf{1} \quad D^*D = \Pi \qquad D(z\mathbf{1} - H(P))^{-1} D^* = P'(z)/N \{P(z)\mathbf{1} - H(P)\}^{-1} \tag{3}$$

where Π is the projection onto the subspace generated by the polynomials of the form p_{nN}, $n \in \mathbf{N}$. If one identifies $L^2(J(P), \mu_p)$ with $l^2(\mathbf{N})$ through the basis given by the p_n's, D appears as the dilation operator $Df(n) = f(Nn)$. The main expected result can be summarized in the following conjecture (this part has been only partially solved in [Bellissard (85b)]):

Conjecture: If P is a monic polynomial of degree N with real coefficients, such that no critical points lie in its Julia set, its Jacobi matrix $H(P)$ is the norm limit of a sequence $H_n(P)$ of periodic Jacobi matrices indexed by $\mathbf{N}$, with periods N^n.

◊

If P is a polynomial such that the conclusion of the previous conjecture is true, we will say that it has the property *LP*. *LP* has been rigorously proven in the following cases [Barnsley (85)]:

Theorem 3: P has the property *LP* in the following cases:
(i) if $P(z) = z^2 - \lambda$ with $\lambda > 3$.
(ii) if $P(z) = a^N T_N(z/a)$ where T_N is the N^{th} Tchebyshev polynomial and $a > \sqrt{3/2}$.
(iii) if $P(z) = a^N T_N(z/a) + b$ where $N = 3$ provided $a \geq 5$, $|b| \leq 5$ or $N = 4$ and $a \geq 2$, $|b| \leq 22$.

◊

Another class of limit periodic operator of interest is given by the so-called "hierarchical models". The first examples were provided by Jona-Lasinio, Martinelli and Scoppola [Jona (84&85)], to illustrate ideas on long range tunneling effect. They got a large class of models with nowhere dense spectra and singular continuous spectral measure. Along the same line Livi, Maritan and Ruffo [Livi] introduced a more specific example given by (1) with:

$$V(0)=0 \qquad V(2^n(2l+1))=v(n) \tag{4}$$

for which one can prove rigorously that the spectrum is nowhere dense with zero Lebesgue measure provided $\limsup_{n->\infty} (v(n+1)-v(n))/(v(n)-v(n-1)) > 2$ [Bellissard (87)].

The most challenging problem is obviously the spectrum of the Almost Mathieu operator $H(\alpha,\mu,x)$ defined on $l^2(\mathbf{Z})$ by (1) with $V(n) = 2\mu \cos 2\pi(x-n\alpha)$. Here μ represents a coupling constant and can be restricted to $\mathbf{R}_+$ without loss of generality; a is a real number but since $H(\alpha+1,\mu,x)=H(\alpha,\mu,x)$ it can be seen as an element of the torus $\mathbf{T}$; x is in $\mathbf{T}$ and represents a generic translation on $\mathbf{Z}$ for it is shifted by α when $H(\alpha,\mu,x)$ is translated by 1 on $\mathbf{Z}$. When α is irrational $H(\alpha,\mu,x)$ is periodic and the usual Bloch or Floquet theory applies. Let $\Sigma(\alpha,\mu)$ be the union over x in $\mathbf{T}$ of the spectra of $H(\alpha,\mu,x)$. We first get:

Theorem 4: (i) If α is irrational, the spectrum of $H(\alpha,\mu,x)$ coincides with $\Sigma(\alpha,\mu)$.
(ii) *Aubry-André's duality:* for every α in $\mathbf{T}$, $\mu\Sigma(\alpha,1/\mu)=\Sigma(\alpha,\mu)$.
(iii) *Aubry-André-Thouless's bound:* the Lebesgue measure of $\Sigma(\alpha,\mu)$ is bounded below by $4|1-\mu|$.

◊

(i) Results from the remark that $H(\alpha+1,\mu,x)$ is unitarily equivalent to $H(\alpha,\mu,x)$ by translation, and is is norm continuous with respect to x. Thus its spectrum is unchanged under the shift $x->x+\alpha$, and is continuous with respect to x. The Aubry-André duality is an argument due to Derrida and Sarma [Derrida] and used by Aubry-André [Aubry (78&80)] to exhibit a metal insulator transition. At last Aubry and André discovered numerically the bound on the Lebesgue measure of $\Sigma(\alpha,\mu)$ and Thouless proved it rigorously [Thouless (83)].

In their original work Aubry and André found also that $\Sigma(\alpha,\mu)$ was a Cantor set whenever α is irrational. This was an extension of the work by Hofstadter on Harper's equation [Harper, Hofstadter] (see fig.2). The earliest rigorous result in this context was given by Bellissard and Simon [Bellissard (82c)]:

Theorem 5: There is a dense G_δ set Ω in $[0,1]\times\mathbf{R}$ such that if $(\alpha,\mu)\in\Omega$ then $\Sigma(\alpha,\mu)$ is nowhere dense.

◊

Actually, one may conjecture that for $\mu \neq 0$ and α irrational the spectrum is nowhere dense. In this result only a generic set of values of α gives this property. This is insufficient for we do not even know whether Ω has positive Lebesgue measure. This theorem has been supplemented by the following result of Sinai [Sinai]:

THEOREM 6: Let α be an irrational number with continued fraction expansion $[a_0, a_1, \ldots, a_n, \ldots]$ such that $a_n \leq \mathrm{const.}\, n^2$. There is $\mu_0 > 0$ such that if $|\mu| > \mu_0$ or if $|\mu| \leq \mu_0$ then the Almost Mathieu operator $H(\alpha, \mu, x)$ has a nowhere dense spectrum of positive Lebesgue measure.

◊

The previous result is in a sense complementary to theorem 5, for the set of α for which theorem 6 holds is the complement of a dense G_δ set but has a full Lebesgue measure.

Another recent result has been provided by Helffer and Sjöstrand [Helffer (87)] using a semiclassical analysis following a renormalization group argument of M. Wilkinson [Wilkinson (84b)]. It concerns specifically the case $\mu=1$, namely the Harper equation.

THEOREM 7: Let E_0 be positive. There is N_0 a positive integer such that for any irrational number α with continued fraction expansion $[a_0, a_1, \ldots, a_n, \ldots]$ such that $a_n \geq N_0$ the spectrum of the Almost Mathieu operator $H(\alpha, \mu=1, x)$ has the following structure:

(i) its convex hull is an interval of the form $[-2+O\ (1/a_1), 2-O\ (1/a_1)]$.

(ii) there is an interval J_0 of length $2E_0+O\ (1/a_1)$ centered at an energy order $O\ (1/a_1)$ such that $SpH(\alpha, \mu, x) \setminus J_0$ is contained in the union of intervals J_i $(N_- \leq i \leq N_+,\ i \neq 0)$ of length $\exp(-C(i)/a_1)$ with $C(i) \approx 1$, separated from each other by a distance of order $O(1/a_1)$.

(iii) for $i \neq 0$ let f_i be the affine increasing map transforming J_i into $[-2,2]$, then $f_i(SpH(\alpha, \mu, x) \cap J_i)$ is contained in the union of intervals $J_{i,k}$ having the same properties as the J_i's provided a_1 be replaced by a_2, and so on.

◊

In this result at each step one has to exclude a central band J_0, $J_{i,0}, \ldots$, in such a way that nothing can be said on the Hausdorff measure of the spectrum which is believed to be 1/2 from numerical calculations [Tang]. On the other hand, the restriction on α is drastic for if $N_0 \neq 1$, it excludes a set of Lebesgue measure one. However it takes into account the self-reproducing properties seen on the Hofstadter spectrum [Hofstadter].

A complementary result on a wider family of quasi periodic operators will be given in section III (Theorems 6&7).

Another interesting class of almost periodic Schrödinger operator with nowhere dense spectrum, is provided by 1D quasicrystals. The first results were provided simultaneously by Kadanoff et al. [Kadanoff] and by Ostlund et al. [Ostlund]. They considered the following model on $l^2(\mathbf{Z})$:

$$H\psi(n) = \psi(n+1) + \psi(n-1) + \mu\chi_{[-\sigma^3,\sigma^2]}(n\sigma)\ \psi(n) \qquad \sigma = \frac{\sqrt{5}-1}{2} \tag{5}$$

Writing the eigenvalue equation $H\psi(n) = E\psi(n)$ in the form $\Psi(n+1) = M(n)\Psi(n)$ where $\Psi(n)$ is the vector in $\mathbf{C}^2$ with components $(\psi(n), \psi(n-1))$ and $M(n)$ is a $2x2$ matrix depending upon E, they showed that if F_n is the n^{th} Fibonacci number defined by $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$, one obtains $A(n) = M(F_n)M(F_n-1)\ldots M(1)$ through the following recursion:

$$A(n+1) = A(n-1)A(n) \tag{6}$$

If now $t(n) = trA(n)$, one easily gets:

$$t(n+2) = t(n+1)t(n) - t(n-1) \tag{7}$$

If $T(n) = (t(n-1),\ t(n),\ t(n+1)) \in \mathbf{R}^3$, (6) is equivalent to $T(n+1) = G(T(n))$, where $G(x,y,z) = (y, z, yz-x)$. A constant of the motion is provided by $I(x,y,z) = x^2+y^2+z^2-xyz$ which defines a hypersurface $\Sigma(E)$ in $\mathbf{R}^3$ depending on the choice of μ and E. By looking at those values of E for which the sequence $t(n)$ is bounded one gets a closed subset of the spectrum of H [Kadanoff, Ostlund (83)]. That it is the full spectrum is a result of Sütö [Sütö]. Remarking that G admits some homoclinic point on $\Sigma(E)$ [Kadanoff], Casdagli [Casdagli] described the spectrum by mean of a Markov partition and a symbolic dynamic to prove that the spectrum is a Cantor set of zero Lebesgue measure and non-zero Hausdorff dimension for $\mu > 8$(this value is probably not the optimal one):

THEOREM 8: Let H be given by (5):
(i) The spectrum of H is given by the set of energies E such that the sequence $t(n) = tr\{M(F_n)M(F_n-1)\ldots M(1)\}$ is bounded.
(ii) The spectrum of H is a Cantor set of zero Lebesgue measure and non-zero Hausdorff dimension for $\mu > 8$.

◊

II-2. *The Metal-Insulator transition:*

In their original work Aubry-André [Aubry (80)] gave an argument on the Almost Mathieu operator to show that a metal insulator transition should occur while the coupling constant varies from $\mu < 1$ to $\mu > 1$. This argument called "Aubry-André's duality" was originally provided by Derrida and Sarma [Derrida] and has been interpreted by Aubry-André in the context of 1D incommensurable chains. Let ψ be a sequence indexed by $\mathbf{Z}$ and solution of the Almost Mathieu equation:

$$\psi(n+1) + \psi(n-1) + 2\mu \cos 2\pi(x-n\alpha)\, \psi(n) = E\, \psi(n) \tag{8}$$

For μ very small, a perturbation argument suggests for ψ an expansion of the form:

$$\psi(n) = e^{2i\pi kn} \sum_{p\in\mathbf{Z}} f(p)\, e^{2i\pi p(x-n\alpha)} \tag{9}$$

Taking this ansatz seriously leads for the $f(p)$'s to the following equation:

$$f(p+1) + f(p-1) + 2/\mu \cos 2\pi(k-p\alpha)\, f(p) = E/\mu\, f(p) \tag{10}$$

We recognize the Almost Mathieu equation after changing μ into $1/\mu$ and rescaling the energy E into E/μ. Suppose that (9) converges say uniformly with respect to x, it follows that the sequence $\{f(p); p\in\mathbf{Z}\}$ is certainly square summable, and that for μ small, the "dual equation" (10) admits E as an eigenvalue. Thus $\mu=1$ is critical and separates a regime where perturbation expansion should in principle be relevant leading to Bloch-like waves, namely extended states, whereas at high coupling the previous "duality" argument gives eigenvalues with localized states. Moreover this argument shows that exponential fall-off of the $f(p)$'s, namely exponential localization, implies analytic dependence of the Bloch waves in the parameter $x-n\alpha$.

To go beyond this heuristic argument, one usually introduces the so-called "Lyapounov exponent" γ representing roughly speaking the rate of exponential increase of a generic solution of (8) at infinity. More precisely let $\Psi=(\psi(0), \psi(1))$ be a vector in $\mathbf{C}^2$, then let ψ be the unique solution of (8) with initial conditions given by Ψ. Then γ is defined by:

$$\gamma(E,\mu,\alpha,x,\Psi) = \limsup_{n\to\infty} \frac{\log\left(|\psi(n+1)|^2 + |\psi(n)|^2\right)}{2n} \tag{11}$$

Proposition 1: Let H be given by (8) with α irrational:
(i) $\gamma(E, \mu, \alpha, x, \Psi)$ is independent of Ψ almost surely (Lebesgue measure).
(ii) $\gamma(E, \mu, \alpha, x)$ is non-negative and independent of x almost surely (Lebesgue measure).
(iii) *Herbert-Jones-Thouless formula:* [Herbert, Thouless (72)]: if $\chi_{[-N,N]}$ is the characteristic function of the interval $[-N,N]$, one has:

$$\gamma(E,\mu,\alpha) = \lim_{N\to\infty} \frac{1}{2N+1} Tr\{\chi_{[-N,N]} \log|E-H|\} \tag{12}$$

(iv) *Aubry-André's duality formula* [Aubry (80)]:

$$\gamma(E, \mu, \alpha) = \log \mu + \gamma(E/\mu, 1/\mu, \alpha) \tag{13}$$

(v) *Aubry-André-Herman's bound* [Aubry (80), Herman]:

$$\gamma(E, \mu, \alpha) \geq \log \mu \tag{14}$$

◊

This set of results can be used for getting an information about the nature of the spectral measure:

THEOREM 9: Let H be the self-adjoint operator given by (8):
(i) *Floquet-Bloch theory:* If α is rational, H has purely absolutely continous spectrum.
(ii) *Pastur-Ishii theorem* [Ishii, Pastur]: If α is irrational, for $\mu > 1$, the absolutely continuous spectrum of H is empty.
(iii) *Delyon's theorem* [Delyon (87)]: If α is irrational, for $\mu < 1$, the point spectrum of H is empty. If $\mu = 1$, the point spectrum if it exists is contained in the set of energies where the Lyapounov exponent vanishes, and the eigenstates are in $l^2(\mathbf{Z})$ but not in $l^1(\mathbf{Z})$.

◊

In the rational case H is periodic and the usual Floquet-Bloch theory applies. In particular the eigensolutions of (8) are Bloch waves of the form given by (9), with an energy $E(k)$ depending analytically on k. In the irrational case, due to the Aubry-André-Herman bound, for $\mu > 1$ the Lyapounov exponent is positive, and the Pastur-Ishii theorem, which is valid for any 1D Schrödinger operator with random potential, implies the absence of absolutely continuous spectrum. The Delyon result is specific to the Almost Mathieu model since it uses Aubry-André's duality in an essential way.

The question is now to know whether for $\mu > 1$ the spectrum is pure point as predicted by Aubry-André's duality. The answer is actually no in general as it follows from the following result by Avron and Simon [Avron (82)]:

THEOREM 10: Let H be given by (8). There is a dense G_δ set Σ of irrational numbers in [0,1] of zero Lebesgue measure such that if $\alpha \in \Sigma$, and $\mu > 1$, H has a purely singular continuous spectrum.

◊

This result is actually a special case of a theorem by Gordon [Gordon] which extends to a wide set of examples. On the other hand, the Σ is contained in the set of "Liouville numbers" namely those irrational numbers for which there is a sequence of rational

p_n/q_n such that $|\alpha - p_n/q_n| \leq 1/q_n^{\,n}$ for all n. These numbers are so rapidly approximated by rationals that the solution of (8) look like Bloch waves on long distances, and never succeed to vanish at infinity.

Conjecture: There is a dense G_δ set Σ of irrational numbers in [0,1] such that if $\alpha \in \Sigma$, and $\mu < 1$, H has a purely singular continuous spectrum.

However one can always argue that Liouville numbers are exceptional since they have Lebesgue measure zero. Almost every number is "diophantine" namely for every $\sigma > 2$, there is $C > 0$ such that $|\alpha - p/q| \geq C/q^\sigma$ for all p/q. Using the Kolmogorov-Arnold-Moser method, Dinaburg and Sinai [Dinaburg] got the existence of some absolutely continuous spectrum with Bloch waves for models given by (1) on $\mathbf{R}$. The adaptation of their technics led Bellissard-Lima-Testard [Bellissard (83a)] to a partial proof of the Aubry-André conjecture, in the sense that only a closed subset of positive Lebesgue measure of the spectrum exhibits a metal insulator transition. This result has been recently supplemented by Fröhlich-Spencer-Wittwer [Fröhlich] and by Sinai [Sinai] which gives:

Theorem 11: Let H be given by (8) and let α satisfy a diophantine condition of the form $|\alpha - p/q| \geq C/q^\sigma$ for all p/q for some $\sigma \geq 3$. Then:

(i) there is $\mu_0 > 0$ such that if $\mu \leq \mu_0$, the absolutely continuous component of the spectrum of H is non empty and is supported by a set of Lebesgue measure bigger than $4 - o(1)$ as $\mu -> 0$.The corresponding eigensolutions have the form (9) with exponentially decaying $f(p)$'s.
(ii) there is $\mu_0 > 0$ such that if $\mu \geq \mu_0$,for almost all x, the spectrum of H is pure point with exponentially localized eigenstates.

◊

The previous result has been extended to various examples on the real line in particular [Dinaburg, Fröhlich]:

Theorem 12: Let H be given by (1) on $\mathbf{R}$.

(i) If $V(x) = \Sigma_{n \in \mathbf{Z}^\nu} v(n)\, exp(\mathbf{i} n.\omega\, x)$ with $\omega \in \mathbf{R}^\nu$ satisfying $|n.\omega| \geq C/|n|^\sigma$ for all $n \in \mathbf{N}^\nu$ and some $C > 0$, $\sigma > \nu$. We suppose $\|V\|_r = \Sigma_{n \in \mathbf{Z}^\nu} |v(n)|\, exp(-r|n|) < \infty$. Then there is E_0 real such that in the interval (E_0, ∞) H admits some absolutely continuous spectrum with eigenfunctions given by Bloch waves of the form $\psi(x) = \exp(\mathbf{i}kx)\ \Sigma_{n \in \mathbf{Z}^\nu} f(n)\, exp(\mathbf{i} n.\omega\, x)$ where the Fourier coefficients $f(n)$ decrease exponentially fast.

(ii) If $V(x) = -\mu\{\cos 2\pi x + \cos 2\pi(\alpha x+\theta)\}$ where α satisfies the diophantine condition $|\alpha - p/q| \geq C/q^3$ for all p/q, μ is large enough, then for almost all q, the spectrum of H in the interval $[-2\mu, -2\mu+O(\sqrt{\mu}\,(1+\alpha^2))]$ is pure point with exponentially localized eigenstates.

◊

It is interesting to note that before these results, nice examples of quasi periodic Schrödinger operators on **Z** have been produced. P. Sarnak [Sarnak] investigated a large class of non self-adjoint operators for which he has been able to compute exactly the spectrum, and found also a transition between pure-point and continuous spectrum. One of the simplest examples of Sarnak operators is given by $H(\mu)\psi(n) = \psi(n+1) + \mu\psi(n)\exp(2\mathbf{i}\pi\alpha n)$. Using a KAM algorithm and an inverse scattering method W. Craig [Craig] produced almost periodic potentials having essentially an arbitrary pure point spectrum. Along the same line Bellissard-Lima-Scoppola [Bellissard (83b)] and Pöschel [Pöschel] exhibited a class of unbounded potentials having dense point spectrum on **R**. This class was derived from the "Maryland model" [Fishman (82)] described by Fishman-Grempel-Prange and which is solvable: it is given by (1) on **Z** with $V(n) = \mu\tan\pi(x-n\alpha)$. It has dense pure point spectrum on **R** if α is diophantine, whereas if α is in some class of Liouville numbers it has singular continuous spectrum [Fishman (83), Simon (84)] (see section II-3 below).

Another question is related to the existence of mobility edges, namely points in the spectrum separating pure point from continuous spectrum. This has been observed numerically by Aubry and André [Aubry (80)], and Bellissard-Formoso-Lima-Testard [Bellissard (82b)] found an almost periodic Schrödinger operator on **R** for which mobility edges do exist. However the corresponding potential is not smooth and the existence of mobility edges for smooth potentials is still an open question.

II-3. *Singular continuous spectra:*

In 1978 Pearson [Pearson] gave an example of Schrödinger operators with a potential vanishing at infinity with purely singular continuous spectrum. For a long time this example was considered as pathological and most of the rigorous results in the literature were concerned with sufficient conditions to avoid singular continuous spectra. In the early eighties, when one started getting results for Schrödinger operators with almost periodic or random potentials, the result of Avron-Simon [Avron (82)] (theorem 10) changed completely the situation and one soon realized that singular continuous spectra were not exceptional, if not the rule for problems related with Solid State Physics. One of the most famous still conjectured example is proved by a 2D Bloch electron in a perfect cubic or hexagonal or triangular crystal submitted to a uniform magnetic field such that the flux through a unit cell is an irrational multiple of the flux quantum: the Hofstadter spectrum is a good example.

The argument of Avron and Simon was based on the remark that 1) in a certain regime, the Lyapounov exponent is positive, which by the Pastur-Ishii theorem prevents absolutely continuous spectrum, and 2) that for a certain class of Liouville numbers the potential is extremely well approximated by periodic potentials (Gordon potentials), which implies by Gordon's theorem [Gordon] the absence of point spectrum. The very same argument applies in various situations. For example in the Maryland model namely the equation (1) on $\mathbf{Z}$ with $V(n) = \tan\pi(x-n\alpha)$, Simon [Simon (84)] defined the quantity $L(\alpha) = \limsup_{n\to\infty} 1/n \log(|\sin(\pi n\alpha)|)$ and proved that if $L(\alpha)=\infty$, the spectrum is purely singular continuous. Fishman-Grempel-Prange [Fishman (83)] investigated the properties of wave functions and found a scale-invariance, showing that they are almost localized on a very sparse sublattice which recurrently reproduces itself at larger scales. The very same argument works as well for a potential of the form $V(n) = 2\mu\cos 2\pi(\alpha n^2+xn+y)$, for both Herman's bound (proposition 1 *(v)*), if α is irrational and Gordon's theorem, if α belongs to a class of Liouville numbers, apply.

In the section II-1. we also introduced the Jacobi matrix of a Julia set, by construction, its spectral measure class is given by the balanced measure on the Julia set. If it is completely disconnected, then one knows that this measure is singular continuous, thus we get another class of singular continuous spectra. For a polynomial of degree 2, $P(z) = z^2-\lambda$, the corresponding Jacobi matrix is given by [Bellissard (82d)]:

$$\begin{array}{lll} H\psi(n) = r(n+1)\psi(n+1)+r(n)\psi(n-1) & n\in\mathbf{N} & \psi(-1)=0 \\ r(0) = 0 \qquad r(2n)^2+r(2n+1)^2 = \lambda & r(2n-1)r(2n)=r(n) & \end{array} \tag{15}$$

THEOREM 13: Let H be given by (15) on $\mathbf{N}$. For $\lambda>2$ the spectrum of H is a Cantor set of zero Lebesgue measure and the spectral measure is purely singular continuous. Any point E in the spectrum can be uniquely labelled by an infinite sequence (or coding) $\underline{\sigma}=(\sigma_0,\sigma_1,\ldots,\sigma_n,\ldots)$ of 0's and 1's, such that $E = \sigma_0(\lambda+\sigma_1(\lambda+\sigma_2\ldots)^{1/2})^{1/2}$. The spectral measure on $\mathbf{R}$ is the image by this map of the Bernoulli measure on the coding. The corresponding eigensolution of $H\psi = E\psi$ satisfies $y_{\underline{\sigma}}(2^k n) = y_{T^k\underline{\sigma}}(n)$ where $\underline{T\sigma}=(\sigma_1,\sigma_2,\ldots,\sigma_{n+1},\ldots)$ and the Lyapounov exponent vanishes on the spectrum.

◊

This result can be extended to any polynomial. It shows in particular that wave functions are not well localized, in contrast with they result of the Maryland group. Moreover they exhibit some chaotic behaviour in space since their value in the large depends upon a random sequence of 0s and 1s.

In the case of 1D quasicrystals Delyon and Petritis [Delyon (86)] proved the following result:

THEOREM 14: Let H be given by (1) on $\mathbf{Z}$ with $V(n) = \mu\chi_A(x-n\alpha)$ where A is an interval on the circle. For Lebesgue almost all α, and any A, the spectral measure of H is purely continuous for Lebesgue almost all x.

◊

Ostlund and Pandit [Ostlund (84)] computed the spectrum of this operator as a function of a and they found a fractal structure suggesting that the Lebesgue measure of the spectrum may be zero. This is an indication that the spectrum may be singular continuous.

At last hierarchical models of Jona-Lasinio, Martinelli and Scoppola [Jona (85)] also give rise to singular spectra. In the case of $\mathbf{Z}$, the class of models described by Li, Maritan, Ruffo [Livi] gives [Bellissard (87)]:

THEOREM 15: Let H be given by (1) on $\mathbf{Z}$ with $V(0) = 0$ and $V(2^n(2k+1)) = v(n)$ for all $k \in \mathbf{Z}$. If $\limsup_{n-\infty} (v(n+1)-v(n))/(v(n)-v(n-1)) > 2$, H has a purely singular continuous spectrum.

◊

These various results show that singular continuous spectra occur normally in many problems of Solid State physics. However Simon et al. [Simon (85&86)] in an argument used for localization gave a result which shows that such spectra are in a certain sense quite unstable under a random perturbation:

THEOREM 16: Let H be a self-adjoint operator having a spectrum supported by a nowhere dense set C of zero Lebesgue measure. Let ψ be a unit vector cyclic for H. Then for Lebesgue almost all μ, the operator $H(\mu) = H+\mu(\psi,.)\,\psi$ has pure point spectrum and the eigenvalues belong to the gaps of C.

◊

This result has been verified for the Jacobi matrix H of the Julia set of a polynomial P by Barnsley-Geronimo-Harrington [Barnsley (85)].

III. *C*Algebras of Almost-Periodic Operators:*

In 1972, Coburn, Moyer and Singer [Coburn] proposed a generalization of the Index formula for elliptic operators on $\mathbf{R}^n$ with almost periodic coefficients. They introduced the C*Algebra of pseudodifferential operators of zeroth order with almost periodic coefficients, and showed that essentially all steps of the usual Index theorem were still valid provided the usual trace of operators be replaced by the trace per unit volume. This idea was exploited later on by Shubin [Shubin] who realized that the integrated density of states of the physicists had a very simple form in this algebraic set-up. In the late seventies, A. Connes generalized the construction to elliptic operators on a foliated compact manifold differentiating along the leaves of the foliation [Connes (82)]. In many cases this C*Algebra admits a natural trace, but there are foliations for which no trace exists. It turns out that most of the problems in Solid State physics involving disordered media, in the independent electrons approximation have hamiltonian affilated to such a C*Algebra [Bellissard (86)]. This has been used to get generic properties of the energy spectrum, such as a gap labelling theorem [Bellissard (82a), (85a), (86), Johnson (82), Delyon (84)], expressions of physical quantities as integrated density of states, Lyapounov exponents, current correlations, for instance. More recently, the definition of a differential structure which is quite natural physically and mathematically, permited to provide a mathematical framework to give a proof of the Quantum Hall Effect [Bellissard (88a&b)] and a detailed study of the Hofstadter spectrum [Bellissard (88b)].

I-1. *Observables and the non-commutative momentum space:*

To start with, let us consider the Almost Mathieu operator. In the section I, eqs. (7, 9, 10) we wrote it in the form:

$$H = U + U^* + \mu(V + V^*) \tag{1}$$

where U and V were two unitaries such that:

$$UV = e^{2i\pi\alpha}\, VU \tag{2}$$

These two operators generate a C*Algebra $\mathcal{A}(\alpha)$ called the rotation algebra. It has been introduced by M. Rieffel [Rieffel] and constitutes a remarkable object in the sense that it is nontrivially non-commutative whenever α is irrational. Nevertheless it is a simple object.

More generally, let H be the Schrödinger operator $H = -\Delta + V$ where V is almost periodic on $\mathbf{R}^D$. The physical system described by H has no longer any translational symmetry and nevertheless it reproduces almost itself under a large translation. On the other hand the translated hamiltonian H_x is equivalent to H in describing the system. Therefore the whole family $\{H_x;\, x \in \mathbf{R}^D\}$ is a natural set of observables. If we insist in performing algebraic calculations, and we need them in practice, we will consider the

C*Algebra generated by $\{H_x;\, x \in \mathbf{R}^D\}$. Since H_x do not commute with H_y for $x \neq y$, this algebra will not be commutative in general. It turns out that this algebra is usually simple to compute in practice [Bellissard (86)]: since V is almost periodic, there is an abelian compact group Ω call the hull of V, a group homomorphism f from $\mathbf{R}^D$ into Ω with a dense image and a continuous function v on Ω such that $V(x) = v(f(x))$. Ω is entirely defined by V or equivalently by H. The C*Algebra $\mathscr{A}$ is then the crossed product $C(\Omega) \times_f \mathbf{R}^D$ of the algebra $C(\Omega)$ of continuous functions on Ω by the action of $\mathbf{R}^D$ defined by f.

A similar treatment can be performed if H is a tight binding approximation, namely an hamiltonian on a lattice like $\mathbf{Z}^D$. It is then sufficient to replace the translation group $\mathbf{R}^D$ by $\mathbf{Z}^D$.

A. Connes developed an analogy with topology or geometry. By Gelfand's theorem (see [Pedersen] for instance), an abelian C*Algebra is isomorphic to the space of continuous functions on a locally compact Hausdorff space vanishing at infinity. Let us decide, by analogy, to identify a non-commutative C*Algebra with the space of continuous functions on some virtual object which will be called a "non-commutative topological space". In our framework, let us consider the special case for which V is periodic. The same construction as before, with the addition of Bloch theory leads to a C*Algebra isomorphic to the tensor product $C(\mathscr{B}) \otimes \mathscr{K}$ where $\mathscr{K}$ is the algebra of compact operators on a separable Hibert space and represents degrees of degenaracy, and $C(\mathscr{B})$ is the space of continuous functions on the Brillouin zone $\mathscr{B}$ (which is usually isomorphic to a torus). Therefore, the periodic case is just the algebra of "functions" over the Brillouin zone (up to the deneracy described by $\mathscr{K}$) which is the crystal analog of the momentum space. By extension, the quasi periodic algebra will be naturally associated to a "non-commutative Brillouin zone". In order that this analogy be efficient, one has to define on these C*Algebras the tools useful in usual geometry: integration theory, differential structure, etc,...

Integration may be obtained through a trace, namely a positive linear (non necessarily bounded) functional τ on $\mathscr{A}$ such that $\tau(AB) = \tau(BA)$ whenever it is defined. It turns out that natural traces can be defined in our situation by mean of a "trace per unit volume". Namely let $d\omega$ be the normalized Haar measure on Ω, which is invariant and (uniquely) ergodic with respect to the action of the translation group $\mathbf{R}^D$. Let also A be an element of $\mathscr{A}$ given by the kernel $a(\omega, x)$ namely acting on $L^2(\mathbf{R}^D)$ through:

$$A_\omega \psi(\mathbf{x}) = \int_{\mathbf{R}^D} d\mathbf{x}'\, a(\omega - f(\mathbf{x}');\, \mathbf{x}-\mathbf{x}')\, \psi(\mathbf{x}') \tag{3}$$

and a similar definition if $\mathbf{Z}^D$ *replaces* $\mathbf{R}^D$. We recall that elements of $\mathscr{A}$ given by smooth kernels are dense in $\mathscr{A}$. The trace per unit volume is then given by:

$$\tau(A) = \lim_{\Lambda \uparrow \mathbf{R}^D} (1/|\Lambda|)\, Tr\{\chi_\Lambda A_\omega) \tag{4}$$

Using the definition (3) of A and the Birkhoff ergodic theorem (see [Halmos] for instance) one gets for almost every ω in Ω:

$$\tau(A) = \lim_{\Lambda \uparrow \mathbf{R}^D} (1/|\Lambda|) \int d\mathbf{x}'\, a(\omega - f(\mathbf{x}'); \mathbf{0}) = \int_\Omega d\omega\, a(\omega; \mathbf{0}) \tag{5}$$

Actually, in the present case, since the action defined by f is uniquely ergodic (namely the Haar measure is the unique ergodic f-invariant probability measure on Ω) the convergence in (5) is uniform with respect to $\omega \in \Omega$. One can easily check that (5) defines a faithful trace on $\mathcal{A}$ which is unbounded in the case of $\mathbf{R}^D$ but bounded and normalized in the case of $\mathbf{Z}^D$.

A natural differential structure can be defined if one remembers that our algebra is supposed to represent functions on the Brillouin zone: differentiating with respect to momentum variables is usually represented in Quantum Mechanics by commutators with the position operators. Let $\mathbf{X} = (X_i)_{i \in \{1,\ldots,D\}}$ be the position operator acting on $L^2(\mathbf{R}^D)$ through:

$$\{X_i \psi\}(\mathbf{x}) = x_i\, \psi(\mathbf{x}) \qquad i = 1,\ldots,D \tag{6}$$

We define derivations ∂_i on $\mathcal{A}$ by:

$$\{\partial_i A\}_\omega = 2\mathbf{i}\pi\, [X_i, A_\omega] \tag{7}$$

or equivalently:

$$\partial_i\, a(\omega, \mathbf{x}) = 2\mathbf{i}\pi\, x_i\, a(\omega, \mathbf{x}) \tag{8}$$

The ∂_i's are linear commuting maps on A, satisfying the fundamental formula of derivations, namely $\partial_i(AB) = (\partial_i A)B + A(\partial_i B)$. Moreover $\tau(\partial_i A) = 0$ whenever the formula makes sense. This allows to get an integration by parts formula $\tau((\partial_i A)B) = -\tau(A\partial_i B)$ showing that usual rules in calculus still hold in this non commutative context.

In [Connes (86)] A. Connes gave also a generalization of line or surface integrals of differential forms. In the commutative context they define a de Rham current. In the non commutative case one may define a closed current as follows: giving $A_0, A_1, \ldots, A_n$ in $\mathcal{A}$, one introduces the formal objects $A_0 dA_1 \ldots dA_n$ by asking that d satisfy the usual rules for a differential, namely $d(AB) = (dA)B + A(dB)$ and $d^2 = 0$; a linear combination of such objects for a fixed n, is called a form of degree n or n-form; let Ω_n be the space of n-forms, and $\Omega(\mathcal{A})$ be the direct sum of the Ω's. One extends the differential d to $\Omega(\mathcal{A})$ by linearity. A closed current is a linear functional τ on $\Omega(\mathcal{A})$ with complex values such that if $\delta\omega$ denotes the degree of ω:

(i) $\tau(\omega_1\omega_2) = (-)^{\delta\omega_1\delta\omega_2}\tau(\omega_2\omega_1)$
(ii) $\tau(d\omega) = 0$
(iii) τ is a current of degree p whenever $\tau(\omega) = 0$ for every $\omega \in \Omega_n$ if $n \neq p$.

As it has been shown by A. Connes in [Connes (86)], a closed current is entirely defined by the $(n+1)$-linear mappings τ_n: $(A_0, A_1,\ldots,A_n) \in \mathcal{A} \to \tau(A_0 dA_1 \ldots dA_n)$ characterized by the following relations:

(a) $$\tau_n(A_1, A_2,\ldots,A_n, A_0) = (-1)^n \tau_n(A_0,\ldots,A_n)$$

(b) $$\sum_{j=0}^{n} (-1)^j \tau_n(A_0,\ldots, A_j A_{j+1},\ldots,A_{n+1}) + (-1)^{n+1}\tau_n(A_{n+1}A_0, A_1,\ldots,A_n) = 0$$

This non commutative differential form theory gives rise to some cohomology namely the Connes cyclic cohomology and a generalization of the abstract index theorem which has already been used partially to the mathematical proof of the Quantum Hall Effect [Bellissard (88a)].

Let us also indicate that besides the previous algebras, one has other physically relevant examples of observable algebras even in the quasi periodic context. For if one considers the situation in which a two dimensional Bloch electron is submitted to a uniform magnetic field B perpendicular to the plane where the electron lies, the hamiltonian is now:

$$H = \frac{1}{2m}\sum_{i=1,2} (P_i - eA_i(\mathbf{X}))^2 + V(\mathbf{X}) \tag{9}$$

where $\mathbf{P}=(P_1, P_2)$ is the momentum operator (namely $P_i = h/2\mathbf{i}\pi\, \partial_i$), $\mathbf{A} = (A_1, A_2)$ is the magnetic vector potential solution of $\partial_1 A_2 - \partial_2 A_1 = B$, m is the electron effective mass, e its electric charge, and V is a periodic potential. The kinetic part is no longer translation invariant because the vector potential breaks the translation symmetry. However adding a phase factor to the translation operator we get the following "magnetic translation" [Zak] on $L^2(\mathbf{R}^2)$:

$$\{U(\mathbf{a})\psi\}(\mathbf{x}) = e^{i\pi eB\mathbf{x}\wedge\mathbf{a}/h}\,\psi(\mathbf{x}-\mathbf{a}) \tag{10}$$

If $\mathbf{a}$ is a period of V, H commutes with $U(\mathbf{a})$. One can then show that the algebra generated by bounded functions of H and its translated is generated by operators A given by a kernel $a(\omega, \mathbf{x})$ defined on $T^2 \times \mathbf{R}^2$ as follows:

$$\{A\psi\}(\mathbf{x}) = \int_{\mathbf{R}^2} d^2\mathbf{x}'\, a(-\mathbf{x}, \mathbf{x}'-\mathbf{x})\, e^{\mathbf{i}\pi eB\mathbf{x}\wedge\mathbf{x}'/h}\,\psi(\mathbf{x}') \tag{11}$$

In much the same way one gets a trace per unit volume and a differential structure on it.

A lattice version of this algebra is precisely given by the rotation algebra we defined in the beginning of this section. The trace and the differential structure are entirely defined by the following conditions:

$$\tau\,(U^m V^n) = \delta_{m,0}\,\delta_{n,0} \tag{12 a}$$

$$\partial_1 U = 2\mathbf{i}\pi U \qquad \partial_1 V = 0 = \partial_2 U \qquad \partial_2 V = 2\mathbf{i}\pi V \tag{12 b}$$

We see that U and V become analogous to the coordinate functions $e^{2\mathbf{i}\pi x_i}$ $(i=1,2)$ of a 2-torus, if the trace is replaced by the normalized Haar measure, if the ∂_i's represent the usual partial derivatives. Because of (2) however this torus is non-commutative.

If one considers the problem of an electron on a quasicrystal submitted to a uniform magnetic field ond will get another kind of algebra which will be hopefully described in a further work [Bellissard (88c)].

III-2. *Gap labelling and K-Theory:*

In the section II we saw that a Schrödinger operator with almost periodic potential has a tendency to exhibit a Cantor spectrum. In particular it must have infinitely many gaps in a bounded interval. The question is whether there is a canonical way of labelling the gaps which is stable under perturbations or under modifications of the frequency module. It happens that this question is related to the computation of integers in the Quantum Hall Effect, and this justifies a complete study. The first gap labelling was provided by Claro and Wannier by a heuristic analysis of the Hofstadter spectrum [Claro (78)]. This labelling was stable under changes of the magnetic field eventhough the spectrum itself is modified in an intricate manner. The first rigorous results came in 1981 with the works of Johnson and Moser [Johnson (82)] and the result of Bellissard-Lima-Testard [Bellissard (82a, 85a, 86)]. A proof in the case of the Almost Mathieu equation was provided by Delyon and Souillard [Delyon (84)]. Johnson and Moser proved it for the case of a 1D Schrödinger operator with an almost periodic potential using ODE technics. But BLT used an algebraic approach namely the K-theory of C*Algebras and got general results valid in any dimension and for any reasonable pseudodifferential operator with almost periodic or even random coefficients [Bellissard (86)]. They used many of the powerful results discovered in the early eighties by the experts in C*Algebras and especially several explicit formular due to A. Connes. It is our aim here to summarize these results.

Let $\mathscr{A}$ be one of the C*Algebras of operators built in the previous section. Let also H be a self-adjoint operator on $L^2(\mathbf{R}^D)$ (or on $l^2(\mathbf{Z}^D)$) bounded from below such that bounded continuous functions of H belong to $\mathscr{A}$. Physicists introduce first the integrated density of states (the IDS) in the following way:

$$\mathcal{N}(\mathrm{E}) = \lim_{\Lambda \uparrow \mathbf{R}^D} (1/|\Lambda|)\ \#\ \{\text{eigenstates of } H|_\Lambda \text{ with energy } \leq E\} \tag{13}$$

where # denotes "the number of", and $H|_\Lambda$ is the operator obtained by restricting H (say in the sense of forms) to a domain D of functions supported by Λ dense in $L^2(\Lambda)$, whenever this makes sense. One can show that because bounded continuous functions of H belong to $\mathcal{A}$, if $\mathcal{A}$ is one of the previous algebras, $H|_\Lambda$ has discrete spectrum bounded from below and therefore the definition of the IDS makes sense. It turns out that the previous formula can be written in a purely algebraic way thanks to the *Shubin formula* [Bellissard (86), Shubin]: if χ_Σ represents the characteristic function of the set Σ, one gets

$$\mathcal{N}(\mathrm{E}) = \tau\ (\chi_{(-\infty,E]}(H)) \tag{14}$$

where τ represents the trace on $\mathcal{A}$ which is extended to the von Neumann algebra generated by $\mathcal{A}$ in the GNS representation of the trace. From this formula it follows that $\mathcal{N}(\mathrm{E})$ is a non decreasing function of E which is constant on the gaps of H. *Thus one way way of labelling the gaps is to affect to it the value of $\mathcal{N}(E)$ for E in this gap.* On the other hand whenever E belongs to a gap of H, $\chi_{(-\infty,E]}(\mathrm{H})$ is actually a continuous and bounded function of H and therefore it belongs to $\mathcal{A}$ and it is also a projection. By Shubin's formula (14) the trace of this projection coincides with the value of the IDS on the gap. This trace actually depends only upon the equivalence class of the projection under unitary transformation. On the other hand $\mathcal{A}$ is a separable C*Algebra and by standard results [Pedersen], the set of such equivalence classes is countable. Therefore the *set of values obtained by taking the traces of projections in $\mathcal{A}$ is a countable subset of the positive real line.* Is it possible to get a rule for its computation?

The answer is actually yes, and the K-theory is the key for it [Atiyah]. For indeed if P is a projection in A, let [P] be its equivalence class as defined by von Neumann, namely the set of projections P' such that there are S and T in $\mathcal{A}$ for which $ST=P$ and $TS=P'$. One can check that if P and Q are orthogonal projections, namely if $PQ=QP=0$, their direct sum $P\oplus Q$ coincides with $P+Q$ and is still a projection in $\mathcal{A}$. Moreover its class $[P\oplus Q]$ depends only upon the classes $[P]$ and $[Q]$ and can be denoted $[P]+[Q]$,defining on the set of classes an addition. This law is not always everywhere defined for it may happen that giving P and Q in $\mathcal{A}$, there is not always a pair $P'\in[P]$ and $Q'\in[Q]$ such that $P'Q'=Q'P'=0$. However, if one enlarges the algebra by taking the C*Algebras $\mathcal{A}\oplus\mathcal{K}$ generated by finite rank matrices over $\mathcal{A}$, one can show that this is always possible to define the sum of two arbitrary equivalence classes. Then by a canonical construction due to Grothendieck one extends this set into an abelian group, which is called $K_0(\mathcal{A})$. If $\mathcal{A}$ is separable this group is countable.

The trace $\tau(P)$ of a projection P in $\mathcal{A}$ has the property that it depends only upon the class $[P]$. Moreover, $\tau(P\oplus Q) = \tau(P)+\tau(Q)$. Therefore it extends into a group homomorphism τ^* from $K_0(\mathcal{A})$ into the real line. From which it follows that (i) the set of real numbers given by the traces of projections in $\mathcal{A}$ is a generating subset of the countable subgroup $\tau^*(K_0(\mathcal{A}))$ of **R,** (ii) the gap labelling defined in this way satisfied sum rules since $\tau^*(K_0(\mathcal{A}))$ is a group. Thus (see [Bellissard (86)]):

Theorem 1 (first gap labelling theorem): If E belongs to gap of H, the IDS $\mathcal{N}(E)$ belongs to the countable subgroup of **R** given by the image $\tau^*(K_0(\mathcal{A}))$ of the K_0-group of $\mathcal{A}$ under the trace homomorphism.

◊

This theorem is actually useless as long as we cannot compute explicitly the K-group. This has been done for the first time for the rotation algebra by Pimsner and Voiculescu [Pimsner] using earlier results of M. Rieffel [Rieffel]:

Theorem 2: Is $\mathcal{A}_\alpha$ is the rotation algebra generated by two unitaries U and V such that $UV=e^{2i\pi\alpha}VU$, its K-group is isomophic to $\mathbf{Z}^2$ and the image $\tau^*(K_0(\mathcal{A}_\alpha))$ of its K-group by the trace homomorphism is $\mathbf{Z}+\alpha\mathbf{Z}$. If P is a projection in $\mathcal{A}_\alpha$ there is a unique integer n such that $\tau(P)=\{n\alpha\}$ where $\{x\}$ denotes the fractional part of x.

◊

The last part of this theorem comes from the remark that since the trace on the rotation algebra is normalized the trace of a projection must belong to the inverval [0,1].

Soon after this result appeared, A. Connes gave a general formula for computing the K-group and its image under the trace homomorphism [Connes (82)]. We will not give it here in detail but we will only give the result one gets in the case of quasi periodic pseudodifferential operators [Bellissard (85a)]:

Theorem 3: Let H be a pseudodifferential operator on $L^2(\mathbf{R}^D)$ with quasiperiodic coefficients. Let $\mathbf{T}^\nu$ be the hull of its coefficients, and let $f(\mathbf{x})_\mu=\Sigma_i\alpha_{\mu i}x_i$ be the corresponding homomorphism with dense image from $\mathbf{R}^D$ into $\mathbf{T}^\nu$. If E belongs to a gap of H, the IDS $\mathcal{N}(E)$ belongs to the subgroup L of **R** given by $L=\Sigma_{(k)}\mathbf{Z}\alpha_{(k)}$ where the $\alpha_{(k)}$'s are the minors of maximal rank of the matrix $\alpha_{\mu i}$. If $D=1$, L coincides with the frequency module of the coefficients of H.

◊

Theorem 4: Let H be a finite difference operator on $l^2(\mathbf{Z}^D)$ with quasiperiodic coefficients. Let $\mathbf{T}^\nu$ be the hull of its coefficients, and let $f(\mathbf{n})_\mu=\Sigma_i\alpha_{\mu i}n_i$ be the corresponding homomorphism with dense image from $\mathbf{Z}^D$ into $\mathbf{T}^\nu$. If E belongs to a gap of H, the IDS $\mathcal{N}(E)$ belongs to the subgroup L of **R** given by $L=\Sigma_{(k)}\mathbf{Z}\alpha_{(k)}$ where the $\alpha_{(k)}$'s are the minors of any rank of the matrix $\alpha_{\mu i}$ including 1 as a minor of rank zero.

If $D=1$, L coincides with $\mathbf{Z}+F$ where F is the frequency module of the coefficients of H.

◊

Various applications of these gap labelling theorems can be found in [Bellissard (86)], especially in connection with the non existence of gaps. Indeed H will have a Cantor spectrum only if the subgroup $\tau^*(K_0(\mathcal{A}))$ is dense in $\mathbf{R}$. Algebraic arguments show that there are examples for which this cannot happen.

In order to illustrate the power of this approach let us however give one example for which the Johnson-Moser argument could not work but the K-theory predicts the result. Consider the operator H_x on $l^2(\mathbf{Z})$ defined by:

$$H_x\psi(n) = \psi(n+1) + \psi(n-1) + \mu\chi_{(0,\beta]}(x-n\alpha)\psi(n) \tag{15}$$

It was shown in [Bellissard (82e)] that the values of the IDS on certain gaps did not follow the rules given by theorem 4 when α and β were rationally independent. Thanks to a recent result of Putnam, Schmidt and Skau [Putnam (85)&(87)] it is possible to compute the K-group of the C*Algebra generated by the H_x's and one finds:

PROPOSITION 1: Let H_x be given by (15) on $l^2(\mathbf{Z})$ where 1, α and β are rationally independent. If E belongs to a gap of H_x, the IDS $\mathcal{N}(E)$ belongs to the countable subgroup $\mathbf{Z}+\mathbf{Z}\alpha+\mathbf{Z}\beta$ of $\mathbf{R}$.

◊

III-3. *Spectrum boundaries:*

As we saw in §I-3, the measurement of the normal metal-superconductor transition curve for a network of superconductors in the temperature magnetic field parameters is equivalent to the measurement of the ground state of the Hofstadter spectrum as a function of the parameter α. This raises the question of computing the spectrum boundaries of a self-adjoint element of the rotation algebra $\mathcal{A}_\alpha$ as a function of α. Unfortunately, the algebras $\mathcal{A}_\alpha$ and $\mathcal{A}_{\alpha'}$ are isomorphic if and only if $\alpha=\pm\alpha'$ (mod. 1). Nevertheless there are many quantities of interest which are obviously continuous functions of this parameter. To overcome this difficulty, one can remark that the family $\{\mathcal{A}_\alpha;\ \alpha\in\mathbf{T}\}$ is a continuous field of C*Algebras in the sense of [Dixmier]. To see this more precisely, let us define the universal rotation algebra $\mathcal{A}$ as the C*Algebra generated by three unitaries U, V, λ, such that:

$$[U,\lambda] = 0 = [V,\lambda] \qquad UV = \lambda VU \tag{16}$$

This algebra is mapped onto $\mathcal{A}_\alpha$ through the *homomorphism ρ_α defined by:

$$\rho_\alpha(U) = U \qquad \rho_\alpha(V) = V \qquad \rho_\alpha(\lambda) = e^{2i\pi\alpha} \tag{17}$$

In much the same way to any closed subset J of $[0,1]$, one associates the algebra $\mathcal{A}(J)$ obtained by restricting the elements of $\mathcal{A}$ to J (the norm on $\mathcal{A}(J)$ satisfies $\|A\| = \sup_{\alpha\in J}\|\rho_\alpha(A)\|$). The next theorem of G. Elliott [Elliott] gives continuity properties with respect to α:

THEOREM 5: Let H belong to the universal rotation algebra. Then:
(i) If $H=H^*$ the gap boundaries of the spectrum of $\rho_\alpha(H)$ are continuous functions of α.
(ii) The norm $\|\rho_\alpha(H)\|$ is a continuous function of α.

◊

This theorem has been supplemented by a theorem of Avron and Simon [Avron (82)]: the spectrum of a Schrödinger operator describing a particle in a uniform magnetic field is continuous with respect to the magnetic field.

It turns out from numerical calculations that the gap boundaries are usually not smooth functions of α (see fig. 2). This has been recently proved by Bellissard (announced in [Bellissard (88b)] following semiclassical ideas developed by Wilkinson [Wilkinson (84a&b)] and Rammal et al. [Wang (87a&b)]. To see this we need further notations.

Let $\mathcal{H}(\mathbf{k}, \alpha)$ be a continuous function of the variables $\mathbf{k} = (k_1, k_2) \in \mathbf{R}^2$ and $\alpha \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. We assume that it satisfies the following properties:

(i) $\mathcal{H}$ is periodic with respect to $\mathbf{k}$ of period 2π in each component of $\mathbf{k}$.
(ii) If $\mathcal{H} = \Sigma_{m\in\mathbf{Z}^2} h(\mathbf{m},\alpha)\, e^{\mathbf{ik}\wedge\mathbf{m}}$ is its Fourier expansion (where $\mathbf{k}\wedge\mathbf{m} = k_1 m_2 - k_2 m_1$) then either:

$$\|\mathcal{H}\|_{(k)} = \sup_{i\le k}\Sigma_{m\in\mathbf{Z}^2} |\partial^i h(\mathbf{m},\alpha)/\partial\alpha^i|(1+|\mathbf{m}|)^k < \infty \qquad \text{for some } k>2 \tag{17a}$$

or the $h(\mathbf{m},\alpha)$'s are holomorphic in α in a strip of width r and

$$|||\mathcal{H}|||_r = \sup_{|Im(\alpha)|\le r}\Sigma_{\mathbf{m}\in\mathbf{Z}^2} |h(\mathbf{m},\alpha)|\, e^{r|\mathbf{m}|} < \infty \qquad \text{for some } r>0 \tag{17b}$$

(iii) For each α in $(-\varepsilon, \varepsilon)$ the function $\mathbf{k} \to \mathcal{H}(\mathbf{k},\alpha)$ has a unique regular minimum in each cell of period. Without loss of generality one can assume that this minimum is located at $\mathbf{k}=0$ for $\alpha=0$ and that $\mathcal{H}(\mathbf{0},0) = 0$.

Correspondingly we define the quantization of $\mathcal{H}$ as the following H element of $\mathcal{A}$:

$$\rho_\alpha(H) = \Sigma_{\mathbf{m}\epsilon\mathbf{Z}^2}\, h(\mathbf{m},\alpha)\; W(\mathbf{m}) \qquad W(\mathbf{m}) = e^{\mathbf{i}\pi\alpha m_1 m_2} U^{m_1}\, V^{-m_2} \tag{18}$$

The ground state energy $E(\alpha)$ is defined as the infimum of the spectrum of $\rho_\alpha(H)$ in $\mathcal{A}_\alpha$. Our first result concerns the asymptotic behaviour of the bottom of the spectrum of $\rho_\alpha(H)$ as $\alpha \rightarrow 0$ namely:

THEOREM 6: Let $\mathcal{H}$ satisfy (i), (ii), (iii) and let H be given by (18). Then there is $E_c > 0$ and $\varepsilon_c \leq \varepsilon$ depending only on $\mathcal{H}$ such that if $\alpha\epsilon\,(-\varepsilon_c, \varepsilon_c)$ the spectrum of $\rho_\alpha(H)$ below E_c is contained in the union of the intervals $\Sigma_n = [E_n(\alpha) - \delta(\alpha), E_n(\alpha) + \delta(\alpha)]$ where if $\mathcal{H}$ satisfies (17a) and δ is equal to $\min(3,k)$:

$$E_n(\alpha) = (2n+1)2\pi\,|\alpha|\,\det{}^{1/2}\{1/2\; D^2.\mathcal{H}(\mathbf{0},0)\} + \alpha\,\partial.\mathcal{H}/\partial\alpha(\mathbf{0},0) + O\,(|\alpha|^{\delta/2}) \tag{19}$$

$$0 < \delta(\alpha) \leq C_1 \mid \alpha\mid^{\delta/2} \tag{20}$$

Here C_1 is a constant depending on $\mathcal{H}$.
If $\mathcal{H}$ satisfies (3b), the estimate (7) is replaced by:

$$0 < \delta(\alpha) \leq \mathrm{C}_1\, e^{-C_2/\alpha} \tag{21}$$

where $C_2 \leq r$.

◊

The proof of this result is a consequence of the semiclassical analysis by Briet-Combes-Duclos [Briet, Combes] and Helffer-Sjöstrand [Helffer (84)]. We then remark that if now $\alpha = p/q\epsilon\mathbf{Q}, \rho_\alpha(H)$ can be computed by mean of the Floquet theory, and admits a band spectrum. If α is close to p/q, the algebra $\mathcal{A}_\alpha$ can be seen as the subalgebra of $M_q \otimes \mathcal{A}_{\alpha-p/q}$ generate by the elements:

$$U_\alpha = W_1 \otimes U_{\alpha-p/q} \qquad V_\alpha = W_2 \otimes \mathrm{V}_{\alpha-p/q} \tag{22}$$

where W_1 and W_2 are qxq unitary matrices such that $W_i^q = \mathbf{1}$ and $W_1 W_2 = e^{2\pi p/q}\, W_2 W_1$. This is a kind of Renormalization Group analysis which was suggested by the work of Wilkinson [Wilkinson (84b)]. Then the limit $\alpha \rightarrow p/q$ can be analysed by using the theorem 6 and the functional calculus to reduce $\rho_\alpha(H)$ on one band of the spectrum. One gets the following result:

THEOREM 7: Let $\mathcal{H}$ satisfy (i), (ii), (iii) above with $\partial h(\mathbf{m},\alpha)/\partial\alpha = 0$, $k \geq 3$, and let B be a non degenerate band of H at $\alpha = p/q$. The lower (resp. the upper) edge of the band $E^-(\alpha)$ (resp. $E^+(\alpha)$) is given by the *Wilkinson Rammal formula:*

$$E^{\pm}(\alpha) = E^{\pm}(\frac{p}{q}) - (\pm)\, a\, |\, \alpha - \frac{p}{q}\, | + b\, (\alpha - \frac{p}{q}) + O\,(\,|\, \alpha - \frac{p}{q}\,|^{3/2}) \tag{23}$$

with:

$$a = \frac{2\pi q^2}{\rho(E^{\pm}(\frac{p}{q}))} \quad ; \quad b = \frac{1}{4\mathbf{i}\pi} tr_q \{ P(\mathbf{k})(\partial_1 H(\mathbf{k})\, \partial_2 P(\mathbf{k}) - \partial_2 H(\mathrm{k})\, \partial_1\, P(\mathbf{k}))\, \}_{E_B(\mathbf{k}) = E^{\pm}(\frac{p}{q})} \tag{24}$$

◊

The first term in this formula is therefore the value of the energy at the band edge for $\alpha = p/q$. The second represents a harmonic oscillator effect, and it produces a discontinuity in the derivate. It shrinks the spectrum in such a way that the neighbouring gap actually increases in size due to this term. The last term comes from a Berry phase, namely from the fact that the eigenprojection $P(\mathbf{k})$ at the value $\alpha = p/q$ defines in general a non trivial line bundle over the 2-torus [Berry, Simon (83)]. This last term accounts for an asymmetry of the derivate around $\alpha = p/q$ and may partially destroy the effect of the first one on the enlargement of the corresponding gap. The derivative of the magnetization of the superconducting array at the transition with respect to the temperature is actually a simple function of the asymmetry of the derivate at each rational point (see [Wang (87a)].

The previous theorem is established for an element H such that $\partial h(\mathbf{m}, \alpha)/\partial\alpha = 0$. If there is $\mathbf{m}$ such that $\partial h(\mathbf{m}, \alpha)/\partial\alpha \neq 0$ one gets an additional contribution to the second term which we will not give here but which is easy to compute.

One consequence of this formula is the following:

Corollary: Let $E(\alpha)$ be a gap boundary for $H \in \mathcal{A}_\alpha$ satisfying (i), (ii), (iii). For any irrational value of α, $E(\alpha)$ is differentiable.

◊

Open problem: Is it possible from this formula to get a proof that the spectrum of H is actually a Cantor set for any irrational α?

Following the strategy of Wilkinson, Helffer-Sjöstrand [Helffer (87)] gave more details in the case of Harper's model, using special positivity properties of its quantization, and their result is the content of the theorem 7 in §II-1.

To finish this section let us indicate that in the previous theorem 7 we used a new type of differential calculus similar to the Ito calculus in stochastic differential equations [Bellissard (88b)]. Namely let A be a polynomial in U, V, λ. One can expand A as follows:

$$\rho_\alpha(A) = \sum_{\mathbf{m} \in Z^2} \mathbf{a}(\mathbf{m}; \alpha)\; W(\mathbf{m}) \qquad \text{with} \qquad W(m_1, m_2) = e^{\mathbf{i}\pi\alpha m_1 m_2} U^{m_1} V^{-m_2} \tag{25}$$

We define the operation ∂ by the following formula:

$$\rho_\alpha(\partial A) = \sum_{\mathbf{m}\in\mathbf{Z}^2} \frac{\partial a(\mathbf{m}:\alpha)}{\partial\alpha} W(\mathbf{m}) \qquad (26)$$

$C^1(\mathcal{A})$ will denote the completion of the set of polynomials under the norm:

$$\|A\|_{C^1} = \|\partial A\| + \|\partial_1 A\| + \|\partial_2 A\| + \|A\| \qquad (27)$$

This operation satisfies the following rules:

THEOREM 8: 1) If A and B belong to $C^1(\mathcal{A})$:

$$\partial(AB) = \partial A\, B + A\partial B + \mathbf{i}/4\pi\, \{\partial_1 A\partial_2 B - \partial_2 A\partial_1 B\} \qquad (28)$$

2) If A belongs to $C^1(\mathcal{A})$ and if it is invertible in $\mathcal{A}$, its inverse belongs to $C^1(\mathcal{A})$ and

$$\partial(A^{-1}) = -A^{-1}\{\partial A + \mathbf{i}/4\pi(\partial_1 A\, A^{-1}\partial_2 A - \partial_2 A\, A^{-1}\partial_1 A)\}\, A^{-1} \qquad (29)$$

◊

The formula (29) is actually the key point in proving (23) & (24) for α close to p/q once they are proved for α small.

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Notes added in proof:

Since the manuscript was written several problems mentioned in the text have been solved:

I. Let us consider the model described in §I-5 (eq.19) with $A = (1-\alpha,1]$ and also in §II-1 (eq.5), §II-2 (theorem 14) and given by the following hamiltonian on $l^2(\mathbf{Z}^2)$:

$$H(x,\alpha)\ \psi(n) = \psi(n+1) + \psi(n-1) + \lambda\ \chi_{(1-\alpha,1]}(x\text{-}n\alpha)\ \psi(n).$$

Its spectrum has been computed numerically by:

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The fractal dimension of the spectrum has been studied non rigorously by:

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In a recent unpublished work, Bellissard, J., Iochum, B., Scoppola, E. & Testard, D., have studied rigorously this model and proved that:

(i) The spectrum of $H(x,\alpha)$ is independent of x for any α's.

(ii) If α is irrational and $\lambda \neq 0$ the spectrum of $H(x,\alpha)$ is a Cantor set.

(iii) The gap boundaries are continuous functions of α as long as α is irrational but they are discontinuous at each rational value of α.

(iv) The spectral measure is purely singular continuous, no states are localized.

(v) The spectrum and the wave functions can be computed from the case $\alpha = 0$ through a renormalization map similar to the map of §II-1 (eq.7) leaving also the same function $I(x,y,z)$ invariant.

II. The nearest neighbours model on a Penrose lattice with or without a magnetic field has been numerically studied in:

[130] Tsunetsugu, H., Fujiwara, T., Ueda, K., Tokohiro, T.: Eigenstates in a 2-dimensional Penrose tiling, *J. of Phys. Soc. Japan*, 55, (1986), 1420-23.

[131] Hatakeyama, T., Kamimura, H.: Electronic properties of a Penrose tiling lattice in a magnetic field, *Solid State Comm.*, 62, (1987), 79-83.

All these works exhibit evidence for Cantor spectrum.

III. Two recent works on the spectrum of the almost Mathieu equation (§II-1) have improved the result of theorem 5:

[132] Van Mouche, P.: The coexistence problem for the discrete Mathieu operator, *to appear in Comm. Math. Phys.*, who proves that the dense G_δ set in theorem 5 is actually independent of the coupling constant as long as it is not zero. And:

[133] Choi, M. D., Elliott, G., Yui, K.: Gauss polynomials and the rotation algebra, *Preprint Swansea* (1988) who give a wonderful proof that in the Harper equation (and also for the Almost Mathieu one if the coupling does not vanish) all the gaps which ought to be open are indeed open when α is rational; as a corollary they get an explicit dense set of irrational numbers for which the spectrum is a Cantor set.

IV. A one dimensional discrete Schrödinger operator with a quasiperiodic potential having two rationally independant frequencies has been studied rigorously by

[134] Sinai, Ya. G.: Anderson localization for the one dimensional difference Schrödinger operator with quasiperiodic potentials, *Proc. Int. Congress Math. Phys. Marseille 1986,* World Scientific, Singapore (1987), pp. 870-903.

[135] Chulaevsky, V. A., Sinai, Ya. G.: Anderson localization for a 1D discrete Schrödinger operator with two-frequency potentials, *subm. to Comm. Math. Phys.* (1988).
It is proved that provided the potential is given by a Morse C^2 function on $\mathbf{T}^3$ and the coupling constant is small enough, there is a set of positive Lebesgue measure in $[0,1]^{\times 2}$ such that for frequencies in that set, the corresponding Schrödinger operator has pure point spectrum with exponentially localized states. On the other hand the spectrum is a connected interval.

V. In a recent unpublished work, S. Kotani proved that if H is the hamiltonian on $l^2(\mathbf{Z}^2)$ given by $H\,\psi(n) = \psi(n+1) + \psi(n-1) + v(n)\,\psi(n)$ where $\mathbf{v} = (v(n))_{n\varepsilon\mathbf{Z}}$ is a sequence with values in a *finite* set, then the spectral measure of H has an absolutely continuous component if and only if the sequence $\mathbf{v}$ is periodic.

VI. The model studied in §II-2 (theorem 15) has recently been investigated non rigorously by

[136] Keirstead, W. P., Ceccatto, H. A., Huberman, B. A.: Vibrational properties of hierarchical systems, *to appear in J. Stat. Phys.* (1988).

VII. The result of J. Avron & B. Simon [Avron (82)] quoted in §III-3 has also been obtained in

[137] Nenciu, G.: Stability of Energy Gaps Under Variations of the Magnetic Field, *Lett. Math. Phys.*, 11,(1986), 127-132.
[138] Nenciu, G.: Bloch electrons in a magnetic field: rigorous justification of the Peierls-Onsager hamiltonian, *Preprint Bucharest*, (1988) (see references therein).